National TEAMS Competition
Wind Energy
Middle School

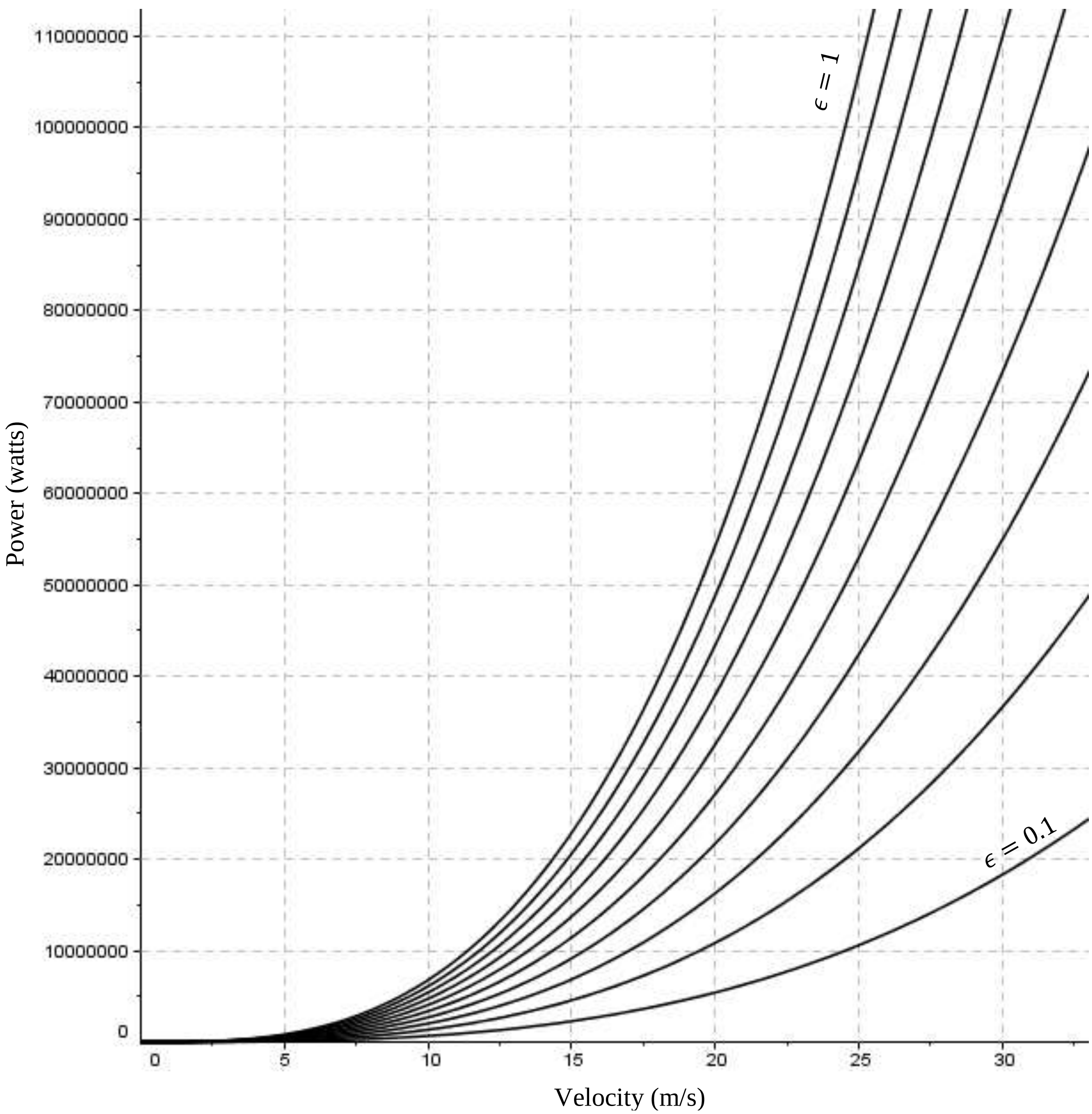
Solutions Manual

1. **Suppose the wind is blowing at a rate of 25 mph through a wind farm consisting of turbines of 120 meters in diameter. We plot the theoretical power output below in increments of 0.1 for ϵ. Using the plot, estimate the input power of the wind assuming the turbine is operating at the Betz limit.**

Given: 25 mph wind >> 11 m/s

Using a value just below ϵ=0.6 gives a value of 5 MW (graphically)

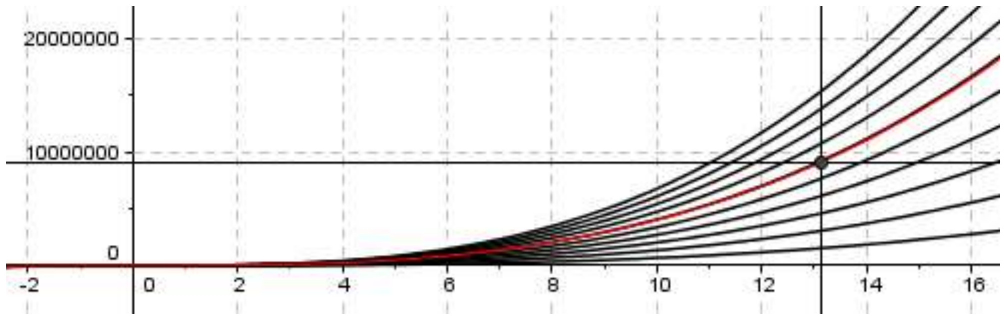
2. **Suppose the wind is blowing at a rate of 30 mph through a wind farm consisting of turbines of 120 meters in diameter. We plot the theoretical power output below in increments of 0.1 for ϵ. Using the plot, estimate the input power of the wind assuming the turbine is operating at the Betz limit.**



Solution: The given information does not match the plot – velocity on the x axis is reported in m/s, not mph. We begin with a conversion.

$$v = 30\, mph\ \times \frac{1\, hour}{3600\, seconds} \times \frac{1609\, meters}{1\, mile} = 13.41\, m/s$$

We now know which x coordinate to reference – this value will need to be approximated by students. Next, we must determine which curve is most relevant to us. Recall that $\epsilon_{Betz} = 0.593$, but we are not given a curve at the Betz limit. The closest curve that corresponds to $\epsilon_{Betz} = 0.593$ is the curve where $\epsilon = 0.6$. Below is the curve with $\epsilon_{Betz} = 0.593$ in red with the point of intersection where $v = 13.41\, m/s$.



We note that the corresponding power is slightly less than 10 MW. Thus, $P_{out} < 10\ MW$. The program being utilized, GeoGebra, claims this value is 9087822.21 W. Next, we will utilize the following relation to calculate the input power:

$$\epsilon_{Betz} = \frac{P_{out}}{P_{in}}$$

$$P_{in} = \frac{P_{out}}{\epsilon_{Betz}}$$

$$P_{in} = \frac{9087822.21 \text{ W}}{0.593} = 1.533 \text{ x } 10^7\ W$$

We were asked to find the mechanical power in horsepower, so we will use the electrical horsepower conversion (1 hp = 550 W).

$$(P_{in})_{hp} = P_{in} \times \frac{1 \text{ hp}}{550\ W} = 27872\ hp$$

Thus, the power input is $\mathbf{2.8\ x\ 10^4\ hp}$. *(Note that, if students assume 10,000,000W the solution is 3.0×10^4, giving the same answer)*

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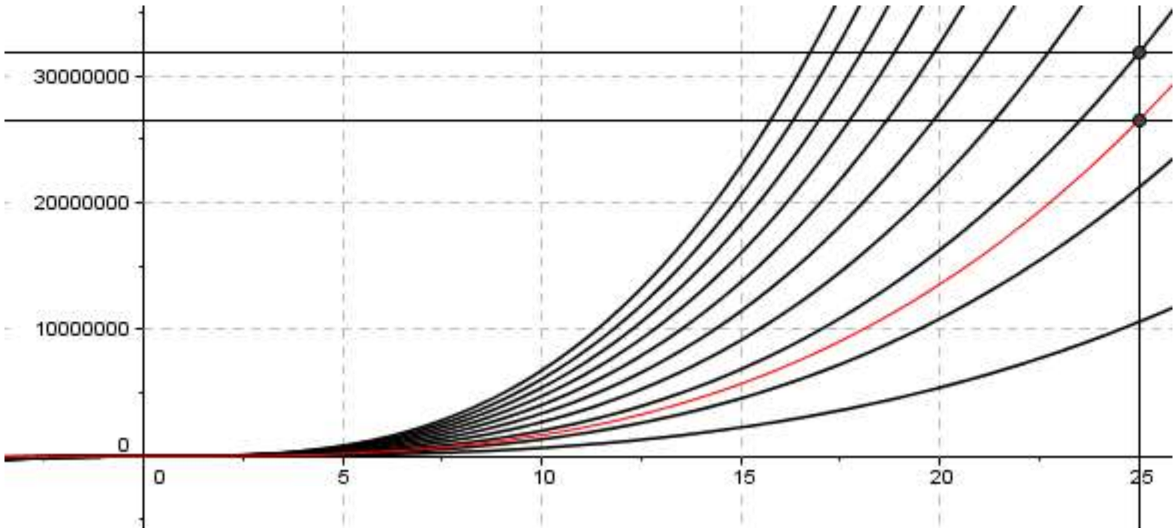
3. The wind begins to accelerate and reaches a rate of 55.92 mph. Given a turbine with typical efficiency, its average expected output power is most nearly:

Solution: We are given that the wind speed is 55.92 mph (or 25 m/s) which enables us to read the x – component of the graph after a conversion. Next, we need to realize that the typical values of epsilon have the following range: $0.25 < \epsilon < 0.30$. While we have a curve with $\epsilon = 0.3$, we will have to interpolate in order to get a curve for $\epsilon = 0.25$ (in red). We know that the curve for $\epsilon = 0.25$ exists halfway between the two curves for $\epsilon = 0.2$ and $\epsilon = 0.3$ respectively. This is easily shown: if $P(v)$ is the general power function, then the power function for $\epsilon = a$ is $P_{\epsilon=a}(v) = aP(v)$.

$$P_{\epsilon=0.25}(x) = \frac{P_{\epsilon=0.20}(v) + P_{\epsilon=0.30}(v)}{2} = \frac{0.20P(v) + 0.30P(v)}{2} = 0.25P(v)$$



We check the associated power with each curve and note that $P_{\epsilon=0.25}(25) = 26507188.01$ and $P_{\epsilon=0.30}(25) = 31808625.62$. Then, the expected power is likely the average of $P_{\epsilon=0.25}(25)$ and $P_{\epsilon=0.30}(25)$. Thus,

$$P_{0.25<\epsilon<0.30} = \frac{P_{\epsilon=0.25}(25) + P_{\epsilon=0.30}(25)}{2} = 29157906.82$$

We then convert this value to electrical horsepower:

$$(P_{0.25<\epsilon<0.30})_{hp} = \frac{P_{0.25<\epsilon<0.30}}{745.7} = 39101.39\ hp$$

which is most nearly **4.0 x 10^4 hp.**

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4. Suppose the blades make a full rotation 15 times per minute. Find the electrical output power if the wind velocity is 34.67 mph.

Solution: We are given angular velocity, ω, in rpm. We will need to convert this to radians per second.

$$\omega = 15\frac{rotations}{minute} \times \frac{2\pi\ radians}{1\ rotation} \times \frac{1\ minute}{60\ seconds} = 1.5708\frac{rad}{s}$$

We know that the blades travel in a circular path, so the velocity will be tangential to the path. Thus,

$$v_t = \omega r$$

The diameter is given in terms of feet - not meters – so will need to make a conversion.

$$r = \frac{393}{2}\ feet\ \times \frac{0.3048\ m}{1\ feet} = 59.893\ m$$

Thus,

$$v_t = \omega r = \left(1.5708 \frac{rad}{s}\right)(59.893\ m) = 94.08 \frac{m}{s}$$

To utilize the lambda v. epsilon curve, we must compute λ.

$$\lambda = \frac{v_t}{v}$$

We are given the velocity in mph, so we must convert to m/s.

$$v = 34.67 \frac{mi}{h} \times \frac{1609.34\ m}{1\ mi} \times \frac{1\ h}{3600\ s} = 15.499 \frac{m}{s}$$

We can then determine lambda:

$$\lambda = \frac{v_t}{v} = \frac{94.08 \frac{m}{s}}{15.499 \frac{m}{s}} = 6.07$$

Referring to the lambda v. epsilon curve, a lambda value of 6.07 (nearly 6) corresponds to an efficiency of 0.4.

$$\lambda = 6.07 \quad \Rightarrow \quad \epsilon = 0.4$$

Returning to the first plot, we then identify the curve associated with $\epsilon = 0.4$. We then go to the corresponding velocity, 15.5 m/s on the *x* axis. We then see that the associated power is nearly 1.0 x 10^7 W or:

$$1.0\ \times 10^7\ W\ \times \frac{1\ hp}{745.7\ W} = \mathbf{1.3\ \times 10^4}\ \boldsymbol{hp}$$

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5. **Consider an arbitrary wind turbine where the tip speed ratio is 5. If the tip of the blade sweeps from 30 degrees to 120 degrees in one second yielding a power of 200 kW, then find the span of the wind turbine blades.**

   Solution: From what we are given, we know that the ratio of the velocities is 5 – the quantity λ. Then, we have:

$$\lambda = \frac{v_t}{v} = 5$$

We need linear velocity to find the tip speed, not angular. Since the path is strictly circular, we know that linear velocity will be tangent to the path. Which means that:

$$v_t = \omega r = \frac{\pi}{2} r \, \frac{m}{s}$$

Thus,

$$\frac{v_t}{v} = \frac{\frac{\pi}{2} r}{v} = 5$$

Which implies:

$$v = \frac{\pi r}{10}$$

Moreover, the radius can be found. From our power relationship:

$$\epsilon = \frac{P_{out}}{P_{in}}$$

$$P_{in} = \frac{P_{out}}{\epsilon}$$

From Figure 1, the efficiency is 30%.

$$P_{in} = \frac{200{,}000}{0.3}$$

From the power formula,

$$P_{in} = \frac{1}{2} \rho A v^3$$

Thus,

$$\frac{200{,}000}{0.3} = \frac{1}{2}(1.23)(2\pi r)\left(\frac{\pi r}{10}\right)^3$$

$$10^3 \left(\frac{200{,}000}{0.3\pi^4(1.23)}\right) = r^4$$

$$r = 48.6\ m$$

The span is the diameter:

$$Span = d = 2r = \mathbf{97.12\ m}$$

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6. The theoretical maximum value is the Betz limit,
Answer is c) 0.593

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7. Find the interval that $f(v)$ is active to satisfy the piecewise definition.

Solution: In order to find the interval that $f(v)$ is active, we need to identify the conditions for the function $C(v)$. Since $C(v)$ is a correction function, its purpose is to gradually increase the power until it reaches its peak efficiency. $f(v)$ will begin when the function is zero and will end when the function is equal to 0.593, the Betz Limit.

$$f(v_{in}) = 0 \quad \text{and} \quad f(v_n) = 0.593$$

We begin by finding the cut-in velocity,

$$f(v_{in}) = f(v) = -0.05(v_{in} - 9)^2 + 1 = 0$$

$$-0.05(v_{in} - 9)^2 + 1 = 0$$

$$(v_{in} - 9)^2 = \frac{1}{0.05}$$

$$v_{in} = 9 \pm \frac{1}{\sqrt{0.05}}$$

$$v_{in} = 4.53\frac{m}{s}$$

Then, we solve $f(v_n) = 0.593$.

$$f(v_n) = -0.05(v_n - 9)^2 + 1 = 0.593$$

$$(v_n - 9)^2 = \frac{0.593 - 1}{-0.05}$$

$$v_n = 9 \pm \sqrt{\frac{0.593 - 1}{-0.05}}$$

$$-\frac{1}{2}(10 - v_n)^2 = \ln(0.593)$$

$$v_n = 10 \mp \sqrt{-2\ln(0.593)}$$

$$v_n = 6.15\frac{m}{s}$$

The intervals given are in mph, so we convert both speeds from m/s to mph. This yields the interval, **[10.13 mph, 13.76 mph).**

8. Find the maximum power generated by the turbine.

Solution: The maximum power occurs at $g(v_n)$. From the equations, we know that:

$$\epsilon = \frac{P_{out}}{P_{in}} = \frac{g(v)}{P_{in}} = \epsilon_{Betz}$$

$$g(v) = P_{in}\epsilon_{Betz}$$

We know the formula for input power.

$$P_{in} = \frac{1}{2}\rho A v^3$$

So, we have a formula for $g(v)$:

$$g(v) = \frac{1}{2}\rho A v^3$$

A is the swept area, so we use the area of a circle: $A = \pi r^2$

$$g(v) = \frac{1}{2}\rho(\pi r^2)v^3$$

We multiply this by $C(v_n)$, which gives us our efficiency at the nominal velocity.

$$g(v)C(v_n) = \frac{1}{2}\rho(\pi r^2)v^3\epsilon_{Betz}$$

Thus,

$$g(v)C(v_n) = 5(1.23)\pi[98]^2v^3(0.593) = 110035v^3$$

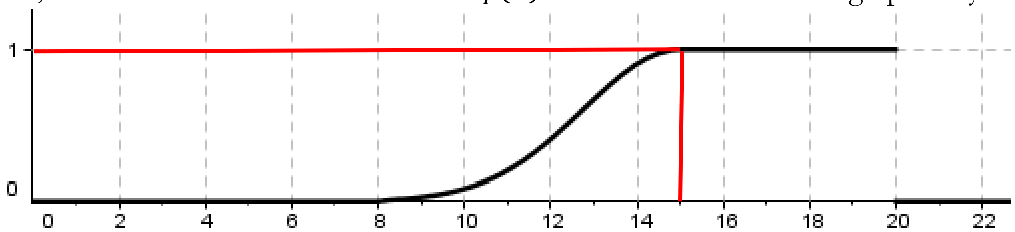
The maximum power occurs at $v_n = 6.15\ m/s$.

$$g(6.15)C(6.15) = 110035(6.15)^3 = \mathbf{2.6 \times 10^7\ W}$$

Let $C_T(v)$ as shown in Figure 3 be used for the turbine. $P(v)$ is the output power for a turbine with a blade length of 98 meters. $P_A(v)$ is the adjusted power.

9. What is the nominal velocity?

Solution: The nominal velocity, v_n, the wind speed which allows the turbine to draw its rated power. Thus, we solve for the first instant that $C_T(v) = 1$. We can check this graphically.



Velocity (m/s)

We see that $C_T(v) = 1$ when $v = 15\ m/s$; however, all of our answers are in mph.

$$v_n = 15\frac{m}{s} \times \frac{3600\ seconds}{1\ hour} \times \frac{1\ mile}{1609\ meters} \cong \mathbf{34\ mph}$$

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10. What is the difference between the ideal power output and the adjusted power output at the cut-out velocity?

Solution: We would like to know the value of the difference, *d*, between the idealized power output (P_I) and the adjusted power output.

$$d = P_I(v_{out}) - P_A(v_{out})$$

We know the output power from our power relationship:

$$P_{out} = P_{in}\epsilon_{Betz} = \frac{1}{2}\rho A v_{out}^3 \epsilon_{Betz}$$

The distinction here is that $P_A(v_{out}) = g(v_n)$, a value that has been fixed since the nominal velocity. $g(v)$ is simply the input power as a function of velocity, so $P_I(v_{out}) = g(v_{out})$. At this point, $C_T(v) = 1$, so

$$d = g(v_{out})C(v_{out}) - g(v_n)C(v_{out})$$

$$d = \frac{1}{2}\rho A v_{out}^3 \epsilon_{Betz} - \frac{1}{2}\rho A v_n^3 \epsilon_{Betz}$$

$$d = \frac{1}{2}\rho A \epsilon_{Betz}(v_{out}^3 - v_n^3)$$

$$d = \frac{1}{2}\rho(\pi r^2)\epsilon_{Betz}(v_{out}^3 - v_n^3)$$

$$d = \frac{1}{2}(1.23)(\pi(98)^2)(0.593)((20)^3 - (15)^3) \cong \mathbf{5.1} \times \mathbf{10^7}\ \boldsymbol{W}$$

Please refer to the following Figure 3 for questions 11 to 13. Figure 3 is the rated power plot for a 3.5 kW wind turbine system that is 33.9%efficient. At the rated power, the blades spin 350 times per minute.

Figure 4: Rated Power for a 3.5 kW Wind Turbine System

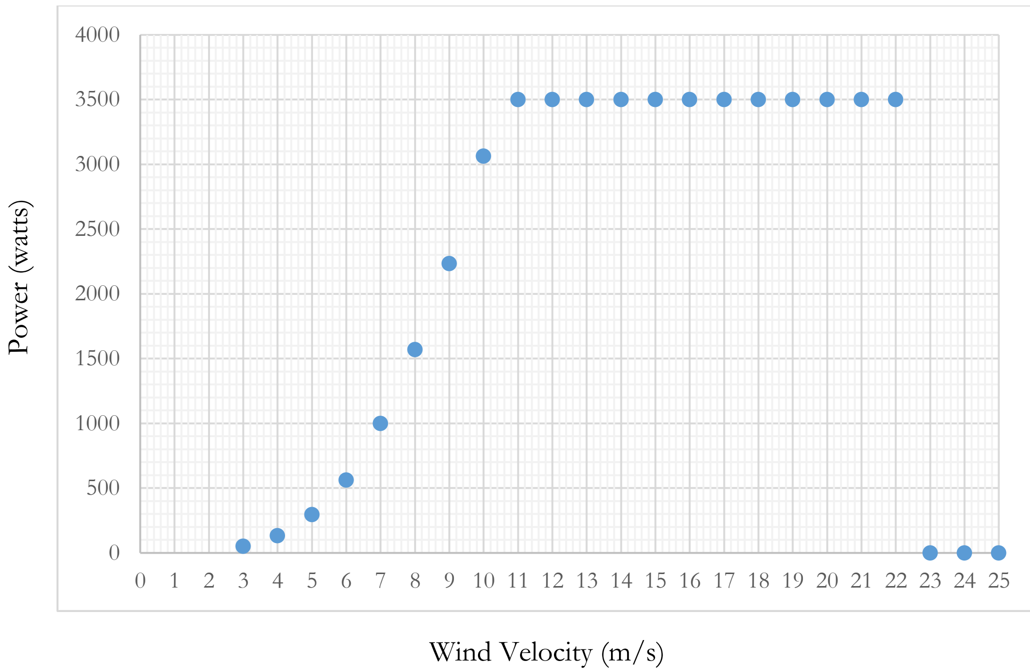

Table 1: Generated Power for a 3.5 kW Wind Turbine System

Wind Speed (m/s)	Output Power (W)	Wind Speed (m/s)	Output Power (W)
1	0	13	3500
2	0	14	3500
3	51	15	3500
4	134	16	3500
5	297	17	3500
6	563	18	3500
7	1000	19	3500
8	1569	20	3500
9	2233	21	3500
10	3064	22	3500
11	3500	23	0
12	3500	24	0

11. How long does it take for the blades to make one revolution?

350 rotations/minute >> 0.00286 minutes/rotation >> 0.171 seconds/rotation

b) 0/17 seconds

12. The best estimate of wind speed for which the turbine reaches its optimal power is

From Table 1, the optimal power occurs at 11 m/s >> 24.6 mph

Best choice: d) 25 mph

13. Find the length of the turbine's blades.

Solution: We know from the power relationship that,

$$\epsilon = \frac{P_{out}}{P_{in}}$$

$$P_{out} = \frac{1}{2}\rho A v^3 \epsilon$$

The area, *A*, can be replaced with πr^2 since the area is circular.

$$P_{out} = \frac{1}{2}\rho(\pi r^2) v^3 \epsilon$$

Solving for the radius yields:

$$r = \sqrt{\frac{2P_{out}}{\rho \pi v^3 \epsilon}}$$

We know the efficiency at the optimal power. In Table 1, we note that the velocity is 11 m/s and the power is 3500 W.

$$r = \sqrt{\frac{2(3500)}{(1.23)\pi(11)^3(0.339)}} = \mathbf{2.00\ m}$$

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14. What is the ratio of velocities associated with the optimal power output?

Solution: We are searching for the ratio of velocities associated with the optimal power output:

$$\lambda = \frac{v_t}{v}$$

From question 11, we know the velocity, but not the tip speed. The tip speed can be rewritten as:

$$v_t = \omega r$$

In the given information, we are told that the blades spin 350 times per minute. Thus,

$$\omega = 350\ rpm$$

but we need this value in radians per second.

$$\omega = 350 \frac{rotations}{minute} \times \frac{2\pi\ radians}{1\ rotation} \times \frac{1\ minute}{60\ seconds} = 36.65 \frac{rad}{s}$$

We then find the tip speed to be:

$$v_t = (36.65)(2) = 73.30$$

Then the ratio can be determined:

$$\lambda = \frac{73.30}{11} \cong \mathbf{6.7}$$

Consider a 1.2 kW turbine that is 28.69% efficient with a ratio of velocities of 12.1481 at its optimal power as shown in Table 2.

Table 2: Generated Power for a 1.2 kW Wind Turbine System

Wind Speed (m/s)	Output Power (W)
1	0
2	0
3	0
4	58
5	127
6	241
7	429
8	673
9	914
10	1200
11	1200
12	1200

15. How many rotations does the turbine blades make per minute?

Solution: We were given that the ratio of velocities is 12.1481 at the optimal power, so that is the lambda value. The optimal power is 1.2 kW and the velocity associated with that power is 10 m/s. We begin with the formula for the ratio of velocities:

$$\lambda = \frac{v_t}{v}$$

We want the amount of times the blades rotates per minute, the angular velocity.

$$v_t = \omega r$$

$$\omega = \frac{v_t}{r}$$

$$\Rightarrow \omega = \frac{v\lambda}{r}$$

Now we must determine the radius:

$$P_{out} = \frac{1}{2}\rho(\pi r^2)v^3\epsilon$$

$$r = \sqrt{\frac{2P_{out}}{\rho\pi v^3\epsilon}} = \sqrt{\frac{2(1200)}{(1.4713)^2\pi(10)^3(0.2869)}} = 1.4713\ m$$

Thus,

$$\omega = \frac{(10)(12.1481)}{(1.4713)} = 82.58\frac{rad}{s}$$

We must convert this to rotations per minute:

$$82.58\frac{rad}{s} \times \frac{1\ rotation}{2\pi\ radians} \times \frac{60\ seconds}{1\ minute} \cong \mathbf{789\ rpm}$$

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16. What is the efficiency of the turbine at the cut-in velocity?

Solution: From looking at the table, we can see that the cut-in velocity is 4 m/s. We found the radius in the previous problem. From the definition of efficiency:

$$\epsilon = \frac{P_{out}}{P_{in}} = \frac{P_{out}}{\frac{1}{2}\rho(\pi r^2)v^3} = \frac{58}{\frac{1}{2}(1.23)(\pi(1.4713\ )^2)(4)^3} \cong 0.22 \ \Rightarrow \mathbf{22\%}$$

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17. **Which relationship between v_1 and v_2 is true given T?**

Solution: According to the fluid flow principles in play with the turbine, the wind speed should decrease upon being disturbed by the turbine. **Thus, $v_1 > v_2$.**

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18. **Suppose the temperature is significantly lower than the standard 15 degrees Celsius. What is a possible consequence of this change?**

Solution: We are now given a scenario where the temperature is decreasing. From Physics, we know that:

$$T_0 \to T \quad \Longrightarrow \quad \rho_0 \to \rho \quad \text{where} \quad T > T_0 \text{ and } \rho_0 < \rho$$

The power available in the wind is dependent on the air density, swept area, and wind speed. At the initial temperature, we have:

$$(P_{in})_0 = \frac{1}{2}\rho_0 A v^3$$

Then at *T*,

$$P_{in} = \frac{1}{2}\rho A v^3$$

At the exact same velocity,

$$\frac{1}{2}\rho A v^3 > \frac{1}{2}\rho_0 A v^3 \quad \Longrightarrow \quad P_{in} > (P_{in})_0$$

From the definition of efficiency, at T_0:

$$\epsilon_{Betz} = \frac{(P_{out})_0}{(P_{in})_0} \quad \Longrightarrow \quad (P_{out})_0 = (P_{in})_0 \epsilon_{Betz}$$

Then,

$$(P_{in})_0 \epsilon_{Betz} < P_{in}\epsilon_{Betz} \quad \Longrightarrow \quad P_{out} > (P_{out})_0$$

This would imply that the output power would increase (**P_{out} increases**)

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19. **Assume *T* is a 750 W turbine with a span of 1 meter and a rated wind speed of 25 mph. What assumption can be made concerning *T*?**

Solution: We are given a seemingly innocent turbine, *T*, let's investigate it a bit. Let's convert the incoming wind speed.

$$v = 25\, mph \times \frac{1\, hour}{3600\, seconds} \times \frac{1609\, meters}{1\, mile} = 12.07\, m/s$$

Next, find the input power,

$$P_{in} = \frac{1}{2}\rho A v^3 = \frac{1}{2}(1.23)(\pi(0.5)^2)(12.07)^3 = 849.35\, W$$

The output power is claimed to be 750:

$$\epsilon = \frac{P_{out}}{P_{in}} = \frac{750}{849.35} = 0.883 > \epsilon_{Betz}$$

We find that the efficiency far exceeds ϵ_{Betz}. Recall that the span is only 1 meter; yet, the output is claimed to be 750 W. In comparison to the other turbines, it may be the case that the **rated power is an overestimate**.

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20. When finding the Betz Limit, we arrive at $\boldsymbol{P_{in} = \frac{1}{2}\rho A v_{avg}(v_1^2 - v_2^2)}$ through the derivation. If we take the ratio of the input power over the undisturbed power of the wind (as if it doesn't travel through a turbine), we have a polynomial in terms of…

Solution: We are given the following**:**

$$\mathrm{P_{in}} = \frac{1}{4}\rho A(v_1 + v_2)(v_1^2 - v_2^2)$$

We ask ourselves what the power would be if the wind was not traveling through the turbine (the undisturbed power). We'll call this power, P_0.

$$P_0 = \frac{1}{2}\rho A v_1^3$$

We are told to form the ratio of the input power over the undisturbed power.

$$P = \frac{P_{in}}{P_0} = \frac{\frac{1}{4}\rho A(v_1 + v_2)(v_1^2 - v_2^2)}{\frac{1}{2}\rho A v_1^3} = \frac{1}{2}\left(\frac{(v_1 + v_2)(v_1^2 - v_2^2)}{v_1^3}\right) = \frac{1}{2}\left(\frac{v_1^3 - v_1 v_2^2 + v_1^2 v_2 - v_2^2}{v_1^3}\right)$$

$$P = \frac{1}{2}\left(1 - \left(\frac{v_2}{v_1}\right)^2 + \left(\frac{v_2}{v_1}\right) - \left(\frac{v_2}{v_1}\right)^3\right)$$

This is a polynomial, $P = \frac{1}{2}(1 + x - x^2 - x^3)$, **in terms of $\boldsymbol{\frac{v_2}{v_1}}$.**
